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Thermal QCD sum rules in the ρ^0 channel revisited

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Abstract. From the hypothesis that at zero temperature the square root of the spectral continuum threshold s_0 is linearly related to the QCD scale Λ we derive in the chiral limit and for temperatures considerably smaller than Λ scaling relations for the vacuum parts of the Gibbs averaged scalar operators contributing to the thermal operator product expansion of the ρ^0 current-current correlator. The scaling with $\lambda \equiv \sqrt{s_0(T)/s_0(0)}$, s_0 being the T-dependent perturbative QCD continuum threshold in the spectral integral, is simple for renormalization group invariant operators, and becomes nontrivial for a set of operators which mix and scale anomalously under a change of the renormalization point. In contrast to previous works on thermal QCD sum rules with this approach the gluon condensate exhibits a sizable T-dependence. The ρ-meson mass is found to rise slowly with temperature which coincides with the result found by means of a PCAC and current algebra analysis of the ρ^0 correlator.

1 Introduction

QCD sum rules at finite temperature have been of intense interest since the pioneering work of Bochkarev and Shaposhnikov [1]. Since thermal field theory lacks asymptotically measurable states the empirical side, that is the spectral function of the QCD sum rule under consideration, must be parameterized. There are few guidelines for the parametrization of the spectral function by hadronic resonance and continuum contributions. However, as pointed out by several authors [2–4], the output of the sum rule at finite temperature or for nonvanishing chemical potential depends on the hadronic model leading to a specific spectral function.

Another complication due to finite temperature arises from the fact, that temperature is defined in a fixed reference frame which hence is singled out. Therefore, the Poincaré invariance of a field theory at $T = 0$ is partially broken for finite temperatures. Residual O(3) and translational invariance permit a wider set of operators to contribute to the operator product expansion (OPE) of the corresponding correlator. Using a background field method, Mallik [5] derived an OPE for the time-ordered thermal correlator of various quark field bilinears, where these new operators are included up to mass-dimension four. In contrast to the work of Hatsuda et. al. [2], which relies on the higher twist classification of operators originating from the analysis of Deep Inelastic Scattering (DIS) and which for nonvanishing spatial momentum components allows for contributions of O(3) non-invariant operators to the thermal OPE[1], a systematic O(3) invariant extension of the zero temperature OPE is obtained in [6]. Thereby the authors consider the mixing of the new operators under a change of the renormalization scale. However, using the background field method, the Wilson-coefficients for the radiative corrections of mass-dimension six could not be calculated, and hence have been omitted in [6]. These contributions are important, since they distinguish the vector from the axial vector channel and cancel a large part of the terms with mass-dimension four, yielding the experimental value of the ρ-meson mass at $T = 0$ with good stability [7].

In [1, 2, 6] the Gibbs average of the OPE is assumed to be saturated by the vacuum and the dilute thermal pion gas contributions. Thereby, both the vacuum and the pion states are taken to be temperature independent. This leads to a T-independent gluon condensate [2], which is in contrast to lattice measurements [8] and to results obtained in effective meson models with scalar glueball fields [9, 10]. The main point of this paper is to explore the consequences of a T-dependent vacuum for the T evolution of the gluon condensate and the ρ-meson mass in the chiral limit. To this end, we parametrize the T-dependence of the vacuum implicitly through the effective spectral variable $s_0(T)$ which divides the hadronic part from the perturbative QCD domain of the spectral function. This ansatz can be justified for renormalization scales of the order of 1 GeV and for temperatures considerably less than the fundamental QCD scale Λ (see Appendix A).

The paper is organized as follows: In Sect. 2 we list the basic results for thermal dispersion relations and for

[1] At first sight this seems to violate rotational symmetry which is retained in the heat bath. Invariance with respect to O(3) transformations is, however, restored in [2] when performing the phase space integrals over pionic matrix elements introduced through the definition of the thermal average.

the spectral function in the ρ^0-meson channel as already established in [2] and [6]. Section 3 contains the thermal OPE for the invariant longitudinal amplitude of the ρ^0-meson correlator, and we briefly discuss the behavior of the nonscalar contributions of mass-dimension four under renormalization. The treatment of thermal operator averages is performed in Sect. 4. With an implicit dependence on temperature, we derive a scaling relation with respect to $\lambda \equiv \sqrt{s_0(T)/s_0(0)}$ for T-dependent vacuum averages of scalar operators. This relation is easily implemented for renormalization group invariants (RGI), and becomes quite involved if one has to regard a set of operators which mix and scale anomalously under a change of the renormalization point. In Sect. 5 we write out the Borel transformed sum rule and, by performing a logarithmic derivative with respect to the inverse squared Borel mass, obtain a sum rule for the ρ-meson mass which is the ratio of two moments. A numerical evaluation of the thermal sum rule is performed in Sect. 6. Section 7 summarizes and compares the results with those of previous approaches.

2 Thermal dispersion relations, kinematic invariants, and spectral function

We consider the thermal correlator of the time-ordered (T) product of two vector currents in the ρ^0-channel

$$T_{\mu\nu}(q,T) = Z^{-1} i \int d^4x\ \mathrm{e}^{iqx} \mathrm{Tr}\ \mathrm{e}^{-\beta H} \mathrm{T} j_\mu(x) j_\nu(0)\ , \quad (1)$$

where

$$j_\mu(x) = \frac{1}{2}(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d) \quad \text{and } Z = \mathrm{Tr}\ \mathrm{e}^{-\beta H}\ . \quad (2)$$

Here H denotes the QCD Hamiltonian, and β stands for the inverse of the temperature T. A sum rule for this correlator can be derived as [6]

$$T_{\mu\nu}(Q_0,|\mathbf{q}|,T) = \frac{1}{\pi}\int_0^\infty dq_0'^{\,2}\ \frac{\mathrm{Im}T_{\mu\nu}(q_0',|\mathbf{q}|)}{q_0'^{\,2}+Q_0^2}\ \tanh(\beta q_0'/2)$$
$$Q_0^2 \equiv -q_0^2\ . \quad (3)$$

The frame of reference where temperature is defined moves with four-velocity u_μ. With this additional covariant one can, for example, define the Lorentz scalars $\omega \equiv u_\mu q^\mu$ and $\bar{q} \equiv \sqrt{\omega^2 - q^2}$. Imposing current conservation and symmetry under exchange of μ and ν, the correlator of (3) can be decomposed as [2,6]

$$T_{\mu\nu}(q,T) = Q_{\mu\nu} T_l(q^2,\omega,T) + P_{\mu\nu} T_t(q^2,\omega,T)\ , \quad (4)$$

where $(g_{\mu\nu}) = \mathrm{diag}(1,-1,-1,-1)$, and the tensors $P_{\mu\nu}$, $Q_{\mu\nu}$ are given by

$$P_{\mu\nu} = -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} - \frac{q^2}{\bar{q}^2}\tilde{u}_\mu\tilde{u}_\nu$$
$$Q_{\mu\nu} = \frac{q^4}{\bar{q}^2}\tilde{u}_\mu\tilde{u}_\nu \qquad \tilde{u}_\mu \equiv u_\mu - \omega\frac{q_\mu}{q^2}. \quad (5)$$

By evaluating $\Pi_1 \equiv u^\mu u^\nu T_{\mu\nu}$ and $\Pi_2 \equiv T^\mu_\mu$ in the rest frame of the heat bath ($u_\mu = (1,0,0,0)$), one can solve for the invariant amplitudes T_l and T_t [6] as

$$T_l = \frac{1}{\bar{q}^2}\Pi_2 \qquad T_t = -\frac{1}{2}\left(\Pi_1 + \frac{q^2}{\bar{q}^2}\Pi_2\right)\ , \quad (6)$$

and in the limit $\mathbf{q}\to 0$ one obtains

$$T_t(q_0,\mathbf{q}=0) = q_0^2\ T_l(q_0,\mathbf{q}=0)\ . \quad (7)$$

Since the sum rule of (3) holds for each component of $(T_{\mu\nu})$, it also holds for Π_1 and Π_2. With (6) one obtains sum rules for the invariant amplitudes T_l and T_t. For example,

$$T_l(q_0^2,\bar{q},T) = \int_0^\infty dq'^{0\,2}\,\frac{N_l(q'^0,\bar{q},T)}{q'^{0\,2}+Q^{0\,2}} \ \text{ with}$$
$$N_l(q'^0,\bar{q},T) \equiv \pi^{-1}\mathrm{Im}T_l\ \tanh(\beta q'^0/2)\ . \quad (8)$$

As already indicated in the introduction, thermal field theory is handicapped by the lack of asymptotically measurable states, and therefore the spectral function (defined as the numerator of the integrand of the right-hand side of (8), that is $N_l(q'^0,\bar{q},T)$), must be modelled. In the following $s_0(T)$ denotes a temperature dependent threshold that divides the hadronic part of the spectrum from the perturbatively accessible QCD domain. It is suggested [1,6], that in the hadronic region of integration, $0 \le q'^{0\,2} \le s_0$, the correlator is saturated by the ρ^0-resonance and the two-pion continuum. Thereby, the pions are assumed to be noninteracting. The hadronic contributions $j_\mu^{\rho^0}$ and j_μ^π to the current j_μ of (2) are obtained by using the field-current-identity

$$j_\mu^{\rho^0} = m_\rho f_\rho \rho_\mu^0 \quad \text{where} \quad f_\rho(T=0) = 153.5\ \mathrm{MeV}\ , \quad (9)$$

and by appealing to the SU(2) flavor symmetry structure resulting in

$$j_\mu^\pi = \varepsilon^{3bc}\pi^b\partial_\mu\pi^c\ . \quad (10)$$

Thereby, the pionic current is obtained by constructing the Noether current of the $\mathrm{SU}(2)_V$ symmetry of a low-energy chiral Goldstone-field theory which to lowest order in the derivatives contains noninteracting fields. Since we only consider a dilute pion gas (one-pion states in Gibbs average) this free pion theory is relevant for our purposes.

Employing the noninteracting, finite temperature (real-time) ρ^0-meson and pion propagators [11–13] for the calculation of the unitarity cuts for the tree diagram of the ρ^0 contribution and for the thermal pion loop, the hadronic part of the spectral function has been calculated in [6]. In contrast to [6] and following [2] we approximate the imaginary part of the correlator above the threshold $s_0(T)$ by perturbative QCD (pQCD). For this perturbative piece thermal contributions are omitted since the fermionic distribution function n_F remains small for the relevant temperature and energy range [2]. In the limit

$\mathbf{q} \to 0$ and after a Borel transformation in Q_0 with Borel mass M [14] the spectral side of the sum rule of (8) reads

$$\begin{aligned} T_l(M,T) &= \frac{1}{\pi M^2}\int_0^\infty ds\ \mathrm{Im}T_l(s,T)\ \mathrm{e}^{-s/M^2} \\ &\quad \times\tanh(\sqrt{s}/(2T)) \\ &\equiv \frac{1}{M^2}\int_0^\infty ds\ \mathrm{e}^{-s/M^2}\ \rho(s,T) \\ &= \frac{1}{M^2}\left(f_\rho^2(T)\mathrm{e}^{-m_\rho^2(T)/M^2} + J_0^{\pi\pi} + J_T^{\pi\pi} + J_0^{qq}\right) \\ &\quad \text{with } s \equiv {q'^0}^2\ , \end{aligned} \tag{11}$$

and

$$\begin{aligned} J_0^{\pi\pi} &= \frac{1}{48\pi^2}\int_{4m_\pi^2}^{s_0(T)} ds \mathrm{e}^{-s/M^2} v^3 \quad J_0^{qq} = \frac{1}{8\pi^2}\left(1+\frac{\alpha_s(\mu)}{\pi}\right) \\ J_T^{\pi\pi} &= \int_{s_0(T)}^\infty ds\ \mathrm{e}^{-s/M^2}\frac{1}{24\pi^2}\int_{4m_\pi^2}^{s_0} ds \\ &\quad (\mathrm{e}^{-s/M^2}v^3 + v(3-v^2)/2)\ n_B(\sqrt{s}/2,T)\ , \end{aligned} \tag{12}$$

where $\mu = 1$ GeV. Here the function v is defined as $v(s,m_\pi) = \sqrt{1-4m_\pi^2/s}$, n_B denotes the Bose distribution, and $J_{0(T)}^{\pi\pi}$, J_0^{qq} are the vacuum (thermal) parts of the spectral integrals due to the $\pi\pi$ and pQCD continua, respectively.

3 Thermal operator product expansion

In this work we use a thermal OPE for the invariant amplitude T_l which combines the results of [6] and [2]. In [6] the expansion is only carried out up to operators of mass-dimension four. The nonscalar O(3) invariant contributions are expressed in terms of diagonal combinations with respect to the anomalous mixing matrix, resulting in a renormalization group invariant (RGI) (that is the total energy density) and a renormalization group non-invariant (RGNI) contribution. The drawback of the OPE of [6] lies in the fact, that mass-dimension six contributions are omitted. These terms are important since they cancel a large part of the nonperturbative correction of mass-dimension four yielding at $T=0$ the experimentally measured ρ-meson mass. In addition, it is the mass-dimension six part of the OPE that distinguishes at $T=0$ the vector from the axial vector channel [7]. Therefore, we use together with the standard $T=0$ scalar operators the results of [2] for the nonscalar mass-dimension six contribution. In the chiral limit, for $\mathbf{q}\to 0$, and for two light quark flavors (u and d)[2], we then obtain the following expansion for the invariant T_l

$$\begin{aligned} &T_l({q^0}^2, \mathbf{q}=0, T) \\ &= -\frac{1}{8\pi^2}\ln\left(\frac{Q_0^2}{\mu^2}\right)(1+\frac{\alpha_s(\mu^2)}{\pi}) + \frac{1}{Q_0^4} \\ &\times\left\{\frac{1}{24}\left\langle\frac{\alpha_s}{\pi}F_{\mu\nu}^a F^{\mu\nu}{}_a\right\rangle_T(\mu) + \frac{2}{11}\left[\langle\theta_{00}\rangle_T(\mu)\right.\right. \\ &\left.\left.+\left(\frac{\alpha_s(\mu^2)}{\alpha_s(Q_0^2)}\right)^{\delta/b}\left\langle\frac{16}{3}\theta_{00}^f - \theta_{00}^g\right\rangle_T(\mu)\right]\right\} \\ &-\frac{\pi}{2Q_0^6}\left\langle\alpha_s\left(\bar{u}\gamma_\mu\gamma_5 t^a u - \bar{d}\gamma_\mu\gamma_5 t^a d\right)^2\right\rangle_T(Q_0) \\ &-\frac{\pi}{9Q_0^6}\left\langle\alpha_s\left(\bar{u}\gamma_\mu t^a u + \bar{d}\gamma_\mu t^a d\right)^2\right\rangle_T(Q_0) \\ &+\frac{8\pi i}{3Q_0^6}\left\langle\bar{u}\gamma_0 D_0 D_0 D_0 u + \bar{d}\gamma_0 D_0 D_0 D_0 d\right\rangle_T(\mu), \end{aligned} \tag{13}$$

where the nonscalar operators are understood to be symmetrized and to be made traceless with respect to the Lorentz indices. The generators t^a of color SU(3) in the fundamental representation are normalized to $\mathrm{Tr}t^a t^b = 2\ \delta^{ab}$. There are, in principle, also O(3) mixed quark-gluon operators of twist 4. However, there is no experimental clue about their matrix elements as pointed out in [2]. There it was also indicated that bag model estimates of the nucleon matrix elements of these operators yield very small values and that the treatment of pions in the bag model is rather questionable due to their collective nature. Following [2] we simply omit these operators to obtain (13).

The Gibbs average in (13) is approximated by the vacuum and the dilute pion gas contributions in [2]. As the authors point out the scalar part of the Gibbs averaged OPE of (13) then has the same structure as the result for the vector correlator obtained in [15]. Based on PCAC, current algebra, and the LSZ reduction formula the authors of [15] find to order T^2 a mixing of the correlator of the vector with that of the axial vector channel due to finite temperature.

To one loop and for two quark flavors the constants b and δ are given by [16]

$$\delta = -\frac{2}{3}\left(\frac{16}{3}+2\right), \quad b = 11 - 2\ \frac{2}{3}\ . \tag{14}$$

At $\mu = 1$ GeV we take $\alpha_s(\mu^2) = 0.36$ [3]. Note that in the case of mass-dimension four the operators a priori renormalized at Q^0 are already expressed by operators evaluated at $\mu = 1$ GeV. Only for $\langle\frac{16}{3}\theta_{00}^f - \theta_{00}^g\rangle_T$ does this process of rescaling generate a renormalization group logarithm due to the nonvanishing anomalous dimension δ of this operator.

The operator $\frac{\alpha_s}{\pi}F_{\mu\nu}^a F^{\mu\nu}{}_a$ is an RGI. In the chiral limit, the fermionic and gluonic parts of the 00-component of the traceless energy-momentum tensor ($\theta_{\mu\nu}$) of one quark flavor QCD [6,17] are

$$\begin{aligned} \theta_{00}^f &= i\bar{q}\gamma_0 D_0 q \\ \theta_{00}^g &= -F_{0\lambda}^a F_0^\lambda{}_a + \frac{1}{4}g_{00}\ F_{\kappa\lambda}^a F^{\kappa\lambda}{}_a \\ D_0 &\equiv \partial_0 - igA_0^a\frac{t_a}{2}\ , \end{aligned} \tag{15}$$

with θ_{00} given by

$$\theta_{00} = \theta_{00}^f + \theta_{00}^g\ . \tag{16}$$

[2] In order to compare our results with the lattice data of [8] for two quark flavors we constrain ourselves to QCD with two light quark flavors throughout the paper.

Since $\theta_{\mu\nu}$ is a conserved quantity the thermal average of θ_{00} is also an RGI. The expression for the covariant derivative D_0 in (15) contains the zeroth component of the gauge potential A_0^a and the color SU(3) generators t_a in the fundamental representation normalized to $\mathrm{Tr} t_a t_b = 2\delta_{ab}$.

In (13) both mass-dimension six scalar operators are RGNI, and we will consider their anomalous scaling and mixing in the next section. As for the O(3), non-scalar mass-dimension six contribution to (13) we neglect the anomalous scaling of the corresponding operator [2]. It is known to belong to a set of operators which mix under renormalization. This set also contains mixed quark-gluon contributions, and following [2] we will omit them.

4 Gibbs averages

4.1 Scalar operators

As was suggested by several authors [1,2,6] the Gibbs averages of the scalar operators of (13) can be saturated by the vacuum and the one pion contributions. Thereby, both the vacuum and the pion states were assumed to exhibit no temperature dependence [2]. In the chiral limit the only quantities entering the sum rule, which can potentially describe thermal pion properties, are the pion decay constant f_π and the pion matrix elements of twist two operators. As for the former the T dependence has been obtained in the imaginary time formulation of thermal chiral perturbation theory in [18]. There, the result for $f_\pi(T)$ is a notably decreasing function of temperature for $T \geq f_\pi$. On the contrary, the lattice simulation of [19] obtains a nearly T independent pion decay constant up to the critical temperature which is understood as an artefact due to the large pion mass (~ 400 MeV) used. As for the T-dependence of the twist two pion averages no information is available so far, and we have to use the pion-in-vacuum parton distributions to estimate the corresponding matrix elements. In accord with previous sum rule investigations at finite temperature [6,2,1] and for consistency we then have no choice but to assume a T-independent pion decay constant.

The motivation for a T-dependent vacuum part in the Gibbs average of scalar operators emerges from a comparison of lattice data for the gluon condensate [8] with the result of [2]. The former approach indicates a sizable decrease above a critical temperature of about $T_c = 140$ MeV, whereas in the latter case the gluon condensate exhibits practically no temperature dependence even when leaving the chiral limit (less than 0.5% decrease at $T = 200$ MeV)[3]. On a more phenomenological level, the T-dependence of the gluon condensate has been obtained from the effective potential of theories based on the non-linear σ-model, where the mesons are coupled to a scalar glueball field to mimic the breaking of scale invariance by the QCD vacuum [9,10]. Depending on the normalization conditions used for the Bag constant and the glueball mass, in these calculations the T-evolution of the gluon condensate exhibits a strong decrease above critical temperatures ranging from $T_c = 140 - 400$ MeV.

The scalar operators appearing in the OPE of the thermal current correlator read

$$\begin{aligned} \dim 4: \quad & \mathcal{O}_4^s = \frac{\alpha_s}{\pi}\, F^a_{\mu\nu} F_a^{\mu\nu} \\ \dim 6: \quad & \mathcal{O}_{6,1}^s = \pi\alpha_s \left(\bar{u}\gamma_\mu\gamma_5 t^a u - \bar{d}\gamma_\mu\gamma_5 t^a d\right)^2 \\ & \mathcal{O}_{6,2}^s = \pi\alpha_s \left(\bar{u}\gamma_\mu t^a u + \bar{d}\gamma_\mu t^a d\right)^2 \ . \end{aligned} \tag{17}$$

Hereby, $\mathcal{O}_4^s$ is an RGI, and $\mathcal{O}_{6,1}^s$, $\mathcal{O}_{6,2}^s$ can be expanded into a basis of scalar four-quark operators $P_1, P_2, ..., P_6$ [14] with

$$\begin{aligned} P_1 &= \bar{\psi}_L\gamma_\mu\psi_L\bar{\psi}_R\gamma_\mu\psi_R \\ P_2 &= \bar{\psi}_L\gamma_\mu t^a\psi_L\bar{\psi}_R\gamma_\mu t^a\psi_R \\ P_3 &= \bar{\psi}_L\gamma_\mu\psi_L\bar{\psi}_L\gamma_\mu\psi_L + (L \to R) \\ P_4 &= \bar{\psi}_L\gamma_\mu t^a\psi_L\bar{\psi}_L\gamma_\mu t^a\psi_L + (L \to R) \\ P_5 &= \bar{\psi}_L\gamma_\mu\lambda^b t^a\psi_L\bar{\psi}_R\gamma_\mu\lambda^b t^a\psi_R \\ P_6 &= \bar{\psi}_L\gamma_\mu\lambda^b\psi_L\bar{\psi}_R\gamma_\mu\lambda^b\psi_R \\ \psi_{L(R)} &= \left(u_{L(R)}, d_{L(R)}\right)^T \ , \end{aligned} \tag{18}$$

where t^a and λ^b are the color SU(3) Gell-Mann and flavor SU(2) Pauli matrices, respectively. They are normalized to $\mathrm{Tr} t^a t^b = \mathrm{Tr}\lambda^a\lambda^b = 2\delta^{ab}$. The left-(right-) handed spinors are given by

$$\psi_{L(R)} = \frac{1}{2}(1 \pm \gamma_5)\psi \ . \tag{19}$$

Using the relation

$$\tau^c_{ij}\tau^c_{mn} = 2(\delta_{in}\delta_{jm} - \frac{1}{N}\delta_{ij}\delta_{mn}) \tag{20}$$

for the SU(N) generator matrices τ^c ($\mathrm{Tr}\tau^a\tau^b = 2\delta^{ab}$) and applying Fierz transformations, one obtains the following decomposition of the flavor singlet parts[4] (for a derivation see the Appendix B)

$$\begin{aligned} \mathcal{O}_{6,1}^s &= \pi\alpha_s \frac{1}{3}\left(-\frac{5}{3}P_4 + \frac{32}{9}P_3 - 2\ P_5\right) \\ \mathcal{O}_{6,2}^s &= \pi\alpha_s \left(P_4 + 2P_2\right) \ . \end{aligned} \tag{21}$$

The sets of operators $P_1, ..., P_4$ and P_5, P_6 mix independently under renormalization with the respective one-loop mixing matrices δ and $\tilde{\delta}$ [20]

[3] On the lattice and at $T = 0$ the ground state average of the local two gluon operator $F^a_{\mu\nu}(0)F_a^{\mu\nu}(0)$ in contrast to the gluon condensate used in an OPE contains also perturbative contributions which must be subtracted for a direct comparison. However, at finite temperature the perturbative part, which contributes to the partition function for momenta larger than, say, 1.5 GeV, is for $T < 200$ MeV severely Boltzmann suppressed and hence hardly influences the T dependence of the thermal average.

[4] Notationally not quite correct we also refer to them as $\mathcal{O}_{6,1}^s, \mathcal{O}_{6,2}^s$.

$$\delta = \begin{pmatrix} 0 & 3/2 & 0 & 0 \\ 16/3 & 17/3 & 0 & -2/3 \\ 0 & -2/3 & 0 & -11/6 \\ 0 & -20/9 & -16/3 & 8/9 \end{pmatrix} \quad \tilde{\delta} = \begin{pmatrix} 7 & 16/3 \\ 3/2 & 0 \end{pmatrix} . \tag{22}$$

The eigenvalues (proportional to the anomalous dimensions of the diagonal combinations) read

$$\begin{aligned} &\delta_1 = 7.043 \;\; \delta_2 = 3.501 \;\; \delta_3 = -2.891 \;, \delta_4 = -1.097 \\ &\tilde{\delta}_5 = 8 \,\text{and}\; \tilde{\delta}_6 = -1 \,. \end{aligned} \tag{23}$$

In Appendix A we argue that for temperatures considerably smaller than the fundamental QCD scale Λ ($\Lambda \approx$ 200 MeV [14,16]), in the chiral limit, and for a renormalization scale Q_0 of the order of 1 GeV it should be possible to describe the T dependence of the thermal average of a scalar operator implicitely through that of the T-dependent spectral continuum threshold $s_0(T)$. Meeting the above premises, we can derive a scaling relation with respect to $\lambda \equiv \sqrt{s_0(T)/s_0(0)}$ for the vacuum average of a given scalar and diagonal operator $\mathcal{O}$ with mass-dimension d and anomalous dimension one-loop coefficient δ_O. In the spirit of a random phase approximation, where correlations between particle-hole excitations in the ground state are included, we set up the QCD vacuum state as an expansion in terms of k-particle (off-shell) quark-antiquark and gluonic fluctuations with vacuum quantum numbers, that is

$$\begin{aligned} |0\rangle &= \sum_k |k\rangle, \;\text{with} \\ |k\rangle &= \sum_{i_1,i_2,...,i_k} \int d^4p_1 \int d^4p_2 \cdots \int d^4p_k \\ &\qquad C_{i_1,i_2,...,i_k}(p_1,p_2,...,p_k)\; |p_1;i_1\rangle \cdots |p_k;i_k\rangle \,. \end{aligned} \tag{24}$$

Hereby, i_j is a collective index labelling the particle species (fermion or boson), spatial quantum numbers, flavor (if fermionic), and color. The $C_{i_1,i_2,...,i_k}$ denote the corresponding expansion coeffcients. Then the vacuum average of $\mathcal{O}$ has a representation of the following form

$$\begin{aligned} \langle 0|\mathcal{O}|0\rangle(Q_0) = \sum_k \sum_{i_1,i_2,...,i_k} \int d^4p_1 \int d^4p_2 \cdots \int d^4p_k \\ f_{i_1,i_2,...,i_k}(p_1,p_2,...,p_k;Q_0) \,, \end{aligned} \tag{25}$$

where the functions $f_{i_1,i_2,...,i_k}$ have mass-dimension $d-4k$, and Q_0 denotes the scale at which the operator $\mathcal{O}$ is renormalized. By asymptotic freedom, the integrand of (25) will be strongly suppressed, if the momenta $p_1,...,p_k$ are hard. The finite value of $\langle 0|\mathcal{O}|0\rangle(Q_0)$ has its origin in the strong dynamics of soft fluctuations in the vacuum [14]. A criterion distinguishing soft from hard fluctuations should roughly be given by the scale s_0. For sizable contributions to the integral of (25) we assume that the time- and space-like virtuality p_j^2 of each of the momenta $p_1,...,p_k$ be less than s_0 and, in addition, that $(p_j^0)^2$ be less then s_0 (recall that through the presence of the heat bath $(p_j^0)^2$ is formally raised to a Lorentz scalar). With the above premises and the results of Appendix A we may assume that temperature effectively acts on the vacuum only implicitely through s_0 and that there is a linear relation between the T dependent QCD scale Λ_T and $\sqrt{s_0(T)}$. Thus there are only two independent scales to be considered: $s_0(T)$ and Q_0. Performing the integrations over the spatial components of the momenta in (25) we obtain

$$\begin{aligned} &\langle 0|\mathcal{O}|0\rangle(Q_0) = \\ &\sum_k \sum_{i_1,i_2,...,i_k} \int_{-\sqrt{s_0(0)}}^{\sqrt{s_0(0)}} dp_1^0 \cdots \int_{-\sqrt{s_0(0)}}^{\sqrt{s_0(0)}} dp_k^0 \\ &h_{i_1,i_2,...,i_k}(p_1^0,p_2^0,...,p_k^0) \times \\ &g_{i_1,i_2,...,i_k}\left(\left\{\frac{p_1^0}{p_k^0},\cdots,\frac{p_{k-1}^0}{p_k^0}\right\},\left\{\frac{p_1^0}{Q_0},\frac{p_1^0}{\sqrt{s_0(0)}}\right\},\cdots,\right. \\ &\left.\left\{\frac{p_k^0}{Q_0},\frac{p_k^0}{\sqrt{s_0(0)}}\right\};\frac{Q_0}{\sqrt{s_0(0)}}\right) \,. \end{aligned} \tag{26}$$

In (26) the functions $h_{i_1,i_2,...,i_k}$ are homogeneous functions of p_i^0, $(i=1,\cdots,k)$ with mass-dimension $d-k$, and $g_{i_1,i_2,...,i_k}$ are dimensionless functions of their dimensionless arguments. The T-dependent vacuum average (not to be confused with the Gibbs average) is then given as

$$\begin{aligned} &\langle 0|\mathcal{O}|0\rangle_T(Q_0) = \\ &\sum_k \sum_{i_1,i_2,...,i_k} \int_{-\sqrt{s_0(T)}}^{\sqrt{s_0(T)}} dp_1^0 \cdots \int_{-\sqrt{s_0(T)}}^{\sqrt{s_0(T)}} dp_k^0 \\ &h_{i_1,i_2,...,i_k}(p_1^0,p_2^0,...,p_k^0) \times \\ &g_{i_1,i_2,...,i_k}\left(\left\{\frac{p_1^0}{p_k^0},\cdots,\frac{p_{k-1}^0}{p_k^0}\right\},\left\{\frac{p_1^0}{Q_0},\frac{p_1^0}{\sqrt{s_0(T)}}\right\},\cdots,\right. \\ &\left.\left\{\frac{p_k^0}{Q_0},\frac{p_k^0}{\sqrt{s_0(T)}}\right\};\frac{Q_0}{\sqrt{s_0(T)}}\right) \,. \end{aligned} \tag{27}$$

With

$$\lambda = \sqrt{\frac{s_0(T)}{s_0(0)}} \,, \tag{28}$$

and from comparison of (26) and (27) one easily obtains

$$\langle 0|\mathcal{O}|0\rangle_T(\lambda Q_0) = \lambda^d \;\langle 0|\mathcal{O}|0\rangle(Q_0) \,. \tag{29}$$

Eq. (29) relates the average of a diagonal operator $\mathcal{O}$ with mass dimension d taken in a temperature destorted vacuum and *renormalized at* λQ_0 to the $T=0$ vacuum average *renormalized at* Q_0. This scaling relation should embody a good approximation for the thermal vacuum average, provided that the above premises of massless quarks, T considerably smaller than Λ, and Q_0 of the order of 1 GeV are fulfilled.

For an RGI operator $\mathcal{O}$ (δ_O=0) the rescaling from λQ_0 to Q_0 is trivial (for example $\mathcal{O}_4^s$). The situation becomes more involved for the scalar operators $\mathcal{O}_{6,1}^s$ and $\mathcal{O}_{6,2}^s$ of mass-dimension six given in (17), and we will focus on them now.

The perturbative one-loop renormalization group rescaling from λQ_0 to Q_0 for the diagonal operator $\mathcal{O}$ is given by

$$\langle 0|\mathcal{O}|0\rangle_T(Q_0) = \left(\frac{\log(\lambda Q_0)^2/\Lambda^2}{\log Q_0^2/\Lambda^2}\right)^{-(\delta_{\mathcal{O}}/b)} \times\langle 0|\mathcal{O}|0\rangle_T(\lambda Q_0) \ , \quad (30)$$

where b is defined in (14). In the OPE, operators renormalized at Q_0 are expressed by operators renormalized at a common reference point μ. Equation (29) implies that first we rescale from λQ_0 to Q_0 which is accomplished by

$$\langle 0|\mathcal{O}|0\rangle_T(Q_0) = \lambda^d \left(\frac{\log(\lambda Q_0)^2/\Lambda^2}{\log Q_0^2/\Lambda^2}\right)^{-(\delta_{\mathcal{O}}/b)} \times\langle 0|\mathcal{O}|0\rangle(Q_0) \ , \quad (31)$$

and then express $\langle 0|\mathcal{O}|0\rangle(Q_0)$ by $\langle 0|\mathcal{O}|0\rangle(\mu)$ as

$$\langle 0|\mathcal{O}|0\rangle_T(Q_0) = \lambda^d \left(\frac{\log(\lambda Q_0)^2/\Lambda^2}{\log Q_0^2/\Lambda^2}\right)^{-(\delta_{\mathcal{O}}/b)} \times \left(\frac{\log Q_0^2/\Lambda^2}{\log \mu^2/\Lambda^2}\right)^{\delta_{\mathcal{O}}/b} \langle 0|\mathcal{O}|0\rangle(\mu) \ . \quad (32)$$

In order to apply the following identity for the Borel transformation $\mathbf{L}_M$ [14]

$$\mathbf{L}_M\left[\left(\frac{1}{Q_0^2}\right)^k \left(\frac{1}{\ln(Q_0^2/\Lambda^2)}\right)^{\varepsilon}\right] = \frac{1}{\Gamma(k)}\left(\frac{1}{M^2}\right)^k \left(\frac{1}{\ln(M^2/\Lambda^2)}\right)^{\varepsilon} \times\left[1+O\left(\frac{1}{\ln(M^2/\Lambda^2)}\right)\right] \ , \quad (33)$$

we expand $\left(\frac{1}{Q_0^2}\right)^k \langle 0|\mathcal{O}|0\rangle_T(Q_0)$ about $\lambda^2 = 1$ up to quadratic order and perform afterwards the Borel transformation indicated in (33). The result is

$$\begin{aligned}
&\mathbf{L}_M\left[\left(\frac{1}{Q_0^2}\right)^k \langle 0|\mathcal{O}|0\rangle_T(Q_0)\right] \\
&\approx \frac{\lambda^{2k}}{\Gamma(k)}\left(\frac{1}{M^2}\right)^k \Big\{\left(\ln(M^2/\Lambda^2)\right)^{\delta_{\mathcal{O}}/b-1} \\
&-(\delta_{\mathcal{O}}/b)\left(\ln(M^2/\Lambda^2)\right)^{(\delta_{\mathcal{O}}/b-2)}(\lambda^2-1) \\
&+\frac{\delta_{\mathcal{O}}}{2b^2}\left[\delta_{\mathcal{O}}+b\left\{1+\ln(\mu^2/\Lambda^2)\right\}\right] \\
&\times\left(\ln(M^2/\Lambda^2)\right)^{(\delta_{\mathcal{O}}/b-3)}(\lambda^2-1)^2 \\
&+O((\lambda^2-1)^3)\Big\}\left(\ln(\mu^2/\Lambda^2)\right)^{(1-\delta_{\mathcal{O}}/b)}\langle 0|\mathcal{O}|0\rangle(\mu) \\
&\equiv \frac{\lambda^{2k}}{\Gamma(k)}\left(\frac{1}{M^2}\right)^k \left\{c_0+c_1(\lambda^2-1)+c_2(\lambda^2-1)^2\right. \\
&\left.+O((\lambda^2-1)^3)\right\}\left(\ln(\mu^2/\Lambda^2)\right)^{(1-\delta_{\mathcal{O}}/b)}\langle 0|\mathcal{O}|0\rangle(\mu) \ . \quad (34)
\end{aligned}$$

With $M^2 \approx 0.75$ GeV^2 and $\Lambda^2 = 0.04$ GeV^2 the coefficient c_2 in the expansion of (34) is roughly given by $1/2c_1$. So even for a value of λ^2 as low as $1/2$ we obtain a suppression for the quadratic as compared to the linear term in (34) by a factor of four.

In the case of the two sets $P_1,...,P_4$ and P_5,P_6 of (18) the mixing under renormalization results in

$$\begin{aligned}
\langle 0|\pi\alpha_s \begin{pmatrix} P_1 \\ \vdots \\ P_4 \end{pmatrix}|0\rangle_T(M) &= \mathbf{T}\mathbf{D}\mathbf{T}^{-1}\langle 0|\pi\alpha_s \begin{pmatrix} P_1 \\ \vdots \\ P_4 \end{pmatrix}|0\rangle(\mu) \\
\langle 0|\pi\alpha_s \begin{pmatrix} P_5 \\ P_6 \end{pmatrix}|0\rangle_T(M) &= \tilde{\mathbf{T}}\tilde{\mathbf{D}}\tilde{\mathbf{T}}^{-1}\langle 0|\pi\alpha_s \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}|0\rangle(\mu)
\end{aligned} \quad (35)$$

where the matrices $\mathbf{T}$ and $\tilde{\mathbf{T}}$, transforming the matrices δ and $\tilde{\delta}$ of (22), respectively, are given by

$$\begin{aligned}
\mathbf{T} &= \begin{pmatrix} 0.1965 & -0.0556 & 0.0496 & 0.6838 \\ 0.9224 & -0.1298 & -0.0955 & -0.5003 \\ -0.0008 & 0.4786 & -0.5481 & 0.3554 \\ -0.3324 & -0.8666 & -0.8295 & 0.3947 \end{pmatrix} \\
\tilde{\mathbf{T}} &= \begin{pmatrix} 8/9 & -16/27 \\ 1/6 & 8/9 \end{pmatrix} \ .
\end{aligned} \quad (36)$$

The diagonal matrices $\mathbf{D} = \mathrm{diag}(D_1,D_2,D_3,D_4)$ and $\tilde{\mathbf{D}} = \mathrm{diag}(D_5,D_6)$ have matrix elements of the form

$$\begin{aligned}
D_i = \lambda^6 &\Big\{\left(\ln(M^2/\Lambda^2)\right)^{(\delta_i/b-1)} - \\
&(\delta_i/b)\left(\ln(M^2/\Lambda^2)\right)^{(\delta_i/b-2)}(\lambda^2-1) + \\
&\frac{\delta_i}{2b^2}\left(\delta_i+b(1+\ln(\mu^2/\Lambda^2))\right)\left(\ln(M^2/\Lambda^2)\right)^{(\delta_i/b-3)} \times \\
&(\lambda^2-1)^2+\mathcal{O}((\lambda^2-1)^3)\Big\} \\
&\left(\ln(\mu^2/\Lambda^2)\right)^{(1-\delta_i/b)} \qquad (i=1,...,6) \ , \quad (37)
\end{aligned}$$

where $\delta_1,...,\delta_6$ are given in (23). Note that the one-loop scaling of α_s is also included in $\mathbf{D}$ and $\tilde{\mathbf{D}}$. When evaluating the sum rule numerically we will consider a truncation of D_i after terms of zeroth and second order in (λ^2-1) to test the sensitivity of the T-evolution of m_ρ, s_0, and the gluon condensate.

Making use of the vacuum saturation hypothesis of Shifman, Vainshtein, and Zakharov (SVZ) [14]

$$\langle 0|\bar{\psi}\Gamma_1\psi\bar{\psi}\Gamma_2\psi|0\rangle = N^{-2}[\mathrm{Tr}\Gamma_1\mathrm{Tr}\Gamma_2 - \mathrm{Tr}\Gamma_1\Gamma_2]\langle 0|\bar{\psi}\psi|0\rangle^2 \ , \quad (38)$$

where $N = 4\times N_f\times N_c$, and the Γ_i $(i=1,2)$, are direct products of Dirac, flavor, and color matrices, one can easily verify [14] that

$$\begin{aligned}
\langle 0|P_2|0\rangle(\mu) &= \frac{16}{3}\langle 0|P_1|0\rangle(\mu) \\
\langle 0|P_3|0\rangle(\mu) &= \langle 0|P_4|0\rangle(\mu) = 0 \\
\langle 0|P_5|0\rangle(\mu) &= \frac{16}{3}\langle 0|P_6|0\rangle(\mu) \ .
\end{aligned} \quad (39)$$

With (21), (35), (37), and (39) we obtain a description for the thermal vacuum contributions of scalar operators to the Gibbs averaged OPE of (13).

As already mentioned in the beginning of this subsection and following [2] we restrict the thermal trace of (1) to the contribution of one-particle pion states (the spectral integral over the pion energy can savely be extended to infinity because of the strong Boltzmann suppression of the integrand). We use the results of [2] which for $\mathcal{O}_4^s$ were obtained by appealing to the trace anomaly of the QCD energy-momentum-tensor. In the chiral limit there is no explicit temperature dependence of the thermal average of the two-gluon operator $\mathcal{O}_4^s$ due to pion contributions. Applying the soft pion theorem twice and using the vacuum saturation hypothesis of (38), according to Hatsuda et al. [2] one obtains for the pionic part of the Gibbs average of the operators $\mathcal{O}_{6,1}^s$ and $\mathcal{O}_{6,2}^s$

$$\sum_{a=1}^{3}\int \frac{d^3p}{2|\mathbf{p}|}\,\langle\pi^a(\mathbf{p})|\frac{1}{2}\mathcal{O}_{6,1}^s+\frac{1}{9}\mathcal{O}_{6,2}^s|\pi^a(\mathbf{p})\rangle\; n_B(|\mathbf{p}|/T) = \frac{128}{81}\pi\alpha_s\langle\bar{q}q\rangle^2\left(1-\frac{3\,T^2}{8f_\pi^2}\right) \,. \tag{40}$$

In the chiral limit we have $f_\pi = 88$ MeV [9], n_B is the Bose distribution, and q indicates a single light flavor quark field. As in [2] we only consider perfect vacuum saturation since for the finite temperature evaluation we are only interested in relative changes as compared to the $T=0$ case.

4.2 O(3) invariant operators

The O(3) invariants of the OPE of (13) have vanishing vacuum expectation values[5] and therefore we only have to consider their pionic matrix elements. In the chiral limit we encountered the following $O(3)$ invariant operators of mass-dimension four in the OPE of (13)

$$\text{dim4}:\quad \mathcal{O}_{4,1}^{O(3)}=\theta_{00}\quad \mathcal{O}_{4,2}^{O(3)}=\frac{16}{3}\theta_{00}^f-\theta_{00}^g\,. \tag{41}$$

The operators $\mathcal{O}_{4,1}^{O(3)}$ and $\mathcal{O}_{4,2}^{O(3)}$ are diagonal combinations with respect to the anomalous mixing matrix of θ_{00}^f and θ_{00}^g [6]. The pion matrix elements of the quark part $N_f\theta_{00}^f$ (N_f denotes the number of quark flavors considered) and the gluon contribution θ_{00}^g to the total energy density θ_{00} are estimated to be equal and can be calculated from the valence parton distribution in the pion as found by Glueck et al. [2,21]. Thereby a low-energy ($Q^2=0.25$ GeV^2) valence like parton distribution fitted to experimental data (direct photon production and Drell-Yan) is evolved at one and two loop order to the sum rule scale of about $Q^2=1.0$ GeV^2. One obtains

$$\sum_{a=1}^{3}\int \frac{d^3p}{2|\mathbf{p}|}\,\langle\pi^a(\mathbf{p})|\theta_{00}^f|\pi^a(\mathbf{p})\rangle(Q)\; n_B(|\mathbf{p}|/T) = \frac{\pi^2T^4}{120}\sum_{a=1}^{3}A_2^{a(u+d)}(Q)\,, \tag{42}$$

with

$$A_2^{a(u+d)}(1\,\mathrm{GeV})=0.972\ \ \forall a\,. \tag{43}$$

At mass-dimension six we consider according to [2] only the following twist two $O(3)$ operator

$$\text{dim6}:\quad \mathcal{O}_6^{O(3)}=i\left(\bar{u}\gamma_0D_0D_0D_0u+(u\to d)\right)\,. \tag{44}$$

Again with the model of Glueck et al. and in the chiral limit the result for the pionic contribution to the Gibbs average of $\mathcal{O}_6^{O(3)}$ has been determined in [2] as

$$\sum_{a=1}^{3}\int \frac{d^3p}{2|\mathbf{p}|}\,\langle\pi^a(\mathbf{p})|\mathcal{O}_6^{O(3)}|\pi^a(\mathbf{p})\rangle(Q)\; n_B(|\mathbf{p}|/T) = -\frac{2}{5}\left(\frac{\pi^4T^6}{63}\sum_{a=1}^{3}A_4^{a(u+d)}(Q)\right)\,, \tag{45}$$

where

$$A_4^{a(u+d)}(1\,\mathrm{GeV})=0.255\ \ \forall a\,. \tag{46}$$

The pion matrix elements of the operators $\mathcal{O}_{4,1}^{O(3)}$, $\mathcal{O}_{4,2}^{O(3)}$, and $O(3)_6$ are of order T^4 or higher which led the authors of [2] to omit them in the sum rule since otherwise higher orders in T would also have to be calculated for the scalar contributions which a priori are of order T^2. We do not agree with this power counting argument. The fundamental approximation for the inclusion of pions in the Gibbs average is that of a dilute pion gas. Higher orders in T would come in by considering interactions between pions as it was shown in [18,22]. Since the pionic matrix elements of the above nonscalar operators happen to be of order T^4 and higher already when interactions are switched off the ommission of these operators would have the effect of artificially introducing pionic interactions. Hence we do not omit the nonscalar operators in the sum rule.

We now have all the ingredients to the Gibbs averaged OPE of (13).

5 Sum rule

Performing a Borel transformation of the OPE of (13), we obtain in the chiral limit the thermal sum rule for the invariant T_l:

$$T_l(M)=\frac{1}{8\pi^2}\left(1+\frac{\alpha_s(\mu)}{\pi}\right)+\frac{1}{M^4}\left\{\left\langle\frac{\alpha_s}{24\pi}F_{\mu\nu}^aF^{\mu\nu a}\right\rangle_T(\mu)\right.$$
$$+\frac{2}{11}\left[\langle\theta_{00}\rangle_T(\mu)+\left(\frac{\alpha_s(\mu^2)}{\alpha_s(M^2)}\right)^{-\delta/b}\right.$$

[5] For diagonal operators, we can, by means of the scaling relation of (29), express their T-dependent vacuum average by a scaling factor times the $T=0$ vacuum average which vanishes due the boost invariance of the vacuum state.

$$\times\left\langle\frac{16}{3}\theta^f_{00}-\theta^g_{00}\right\rangle_T(\mu)\Bigg]\Bigg\}-\frac{1}{M^6}\left(\frac{1}{12}\left\langle-\frac{5}{3}P_4+\frac{32}{9}P_3-2\;P_5\right\rangle_T(M)+\frac{1}{18}\langle P_4+2\;P_2\rangle_T(M)+\frac{4\pi i}{3}\langle\bar{q}\gamma_0 D_0 D_0 D_0 q\rangle_T(\mu)\right)\;, \tag{47}$$

where the dispersion integral for $T_l(M)$ appearing on the left-hand side of (47) is given by (12) and (11). The scalar mass-dimension six operators P_1, P_3, P_4, P_6 renormalized at M can be expressed by the operators P_2 and P_5 renormalized at $\mu = 1$ GeV with the help of (35) and (39).

In order to eliminate the coupling f_ρ we solve (47) for the ρ-meson term

$$R \equiv m_\rho^2 \mathrm{e}^{-m_\rho^2/M^2}\;,$$

substitute $\tau \equiv 1/M^2$, and perform the logarithmic derivative of R [23,24] to obtain the ratio of moments

$$m_\rho^2(T, s_0, \tau) = -\frac{\partial}{\partial\tau}\log\, R(T, s_0, \tau) \tag{48}$$

which we will consider.

6 Numerical evaluation

In this section we discuss the numerical evaluation of the sum rule of (48). Thereby, we employ the following values for the one-flavor quark and the gluon condensate [3]:

$$\langle 0|\bar{q}q|0\rangle\;(\mu = 1\text{ GeV}) = -(250\text{ MeV})^3$$
$$\langle 0|\frac{\alpha}{\pi}F^a_{\mu\nu}F^{\mu\nu}_a|0\rangle\;(\mu = 1\text{ GeV}) = 0.012\text{ GeV}^4\;. \tag{49}$$

As a result of vacuum sum rule calculations in a variety of channels the central value for the gluon condensate is actually believed to be twice as high as the value stated in (49) with an error of about 50%. The central value for the quark condensate indicated here is rather up to date and exhibits a smaller error of about 30% [25]. However, the higher values for the gluon condensate have been obtained using numerical correction factors κ of about $\kappa = 2$ which multiply the four-quark condensate derived from the vacuum saturation hypothesis [14]. For exact vacuum saturation ($\kappa = 1$) the values of (49) for the quark and gluon condensates nicely reproduce the ρ-meson mass in the vacuum ($m_\rho(T = 0) \approx 760$ MeV). Since we are only interested in relative changes of the spectral parameters and condensates for $T > 0$ as compared to the zero temperature case we will stick to the values of (49) [2].

Our strategy in evaluating the sum rule of (48) is as follows:

Since τ is not a direct physical observable, it is chosen as the stationary point τ_s of $m_\rho^2(0, s_0(0), \tau)$ for a given s_0 at $T = 0$.

For a value of $s_0(0) = 1.5$ GeV^2 an additional contribution to the spectral function due to $a_1 - \pi$ production is regarded as a part of the pQCD continuum. This is reasonable as long as the Borel parameter τ is larger than 1 GeV^{-2} since structures stemming from the $a_1 - \pi$ production and possible radial excitations of the ρ-meson are then sufficiently suppressed in the spectral integral [26].

There is a pronounced minimum for a numerically obtained value of $M_s^2 \equiv 1/\tau_s \approx 0.73$ GeV^2 corresponding to a ρ-meson mass of 745 MeV in the chiral limit considered here.

For finite temperatures T we determine the value of $s_0(T)$ from the stationary point of $m_\rho^2(T, s_0(T), \tau)$ at the same value $\tau = \tau_s$ as in the zero temperature case. Performing this calculation at a sufficient number of T points yields the temperature evolution of m_ρ^2, s_0 and, with the parametrization of (29), also the temperature dependence of the gluon condensate. The usefulness of the procedure to determine the T-evolution for a fixed τ_s was checked by applying this technique to the ρ^0 sum rule given in (4.4) of [2] ($s_0(0) = 1.5$ GeV^2). Our method yields the same results as found in [2], where a more elaborate evaluation analysis (averaging over a T-dependent Borel window) was used.

We consider the following two cases: (a) naive rescaling[6] and (b) renormalization group rescaling when including terms up to quadratic order in $\lambda^2 - 1$ in the expansion of D_i given in (37).

The temperature evolution is determined in steps of $\Delta T = 3$ MeV. For $s_0(0) = 1.5$ GeV^2 and in the case (b) the ρ-meson mass exhibits an increase of about 17.5% from its zero temperature value at a 'critical temperature' of $T_c = 157$ MeV which is in contrast to the result of [2]. Thereby, T_c is obtained by demanding the following: If a variation of $s_0(T)$ with respect to $s_0(T-\Delta T)$ is larger than 0.3 GeV^2 to produce the minimum of the Borel curve for m_ρ at τ_s, we set $T = T_c$. This critical value merely indicates the breakdown of our sum rule analysis and may not coincide with the critical temperature of a deconfinement phase transition. There is no strong dependence of T_c on the initial value of s_0 for the case (b) – $T_c(s_0(0) = 1.2$ GeV^2)= 157 MeV, $T_c(s_0(0) = 1.5$ GeV^2)=157 MeV, and $T_c(s_0(0) = 1.8$ GeV^2)=163 MeV. Figure 1 shows the results for m_ρ as a function of the temperature, where for $s_0(0) = 1.2$ GeV^2 and $s_0(0) = 1.8$ GeV^2 only the case (b) has been considered. There is practically no difference in the evolution of m_ρ when including the quadratic terms in $\lambda^2 - 1$ for the D_i of (37) as compared to the linear and to the zeroth order approximation. The deviation at $T = 160$ MeV is then at most 5 MeV.

Figure 2 indicates the temperature dependence of the pQCD threshold $s_0(T)$ for the three initial values and the case (b). In addition, we show the case (a) choosing $s_0(0) = 1.5$ GeV^2. Again, there is practically no difference between the results for truncations of D_i in zeroth, linear, and quadratic order. Hence the logarithmic corrections in the Borel parameter τ seem to have a much

[6] Naive rescaling means, that, according to (37), the anomalous renormalization group rescaling of the operators (powers of logarithms) are suppressed, leaving only the factor λ^6 at mass-dimension six.

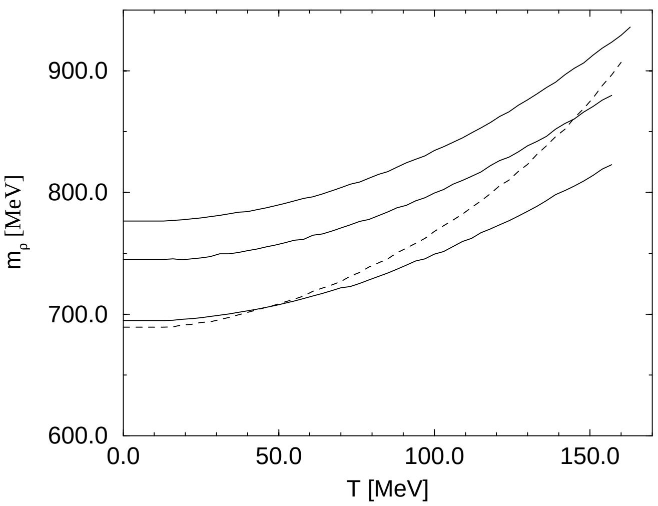

Fig. 1. Temperature evolution of the ρ-meson mass. The solid lines correspond to the case (b) for $s_0(0) = 1.2$ GeV^2 ($m_\rho(T = 0) = 694$ MeV), $s_0(0) = 1.5$ GeV^2 ($m_\rho(T = 0) = 745$ MeV), and $s_0(0) = 1.8$ GeV^2 ($m_\rho(T = 0) = 776$ MeV). For $s_0(0) = 1.5$ GeV^2 the case (a) is associated with a dashed line

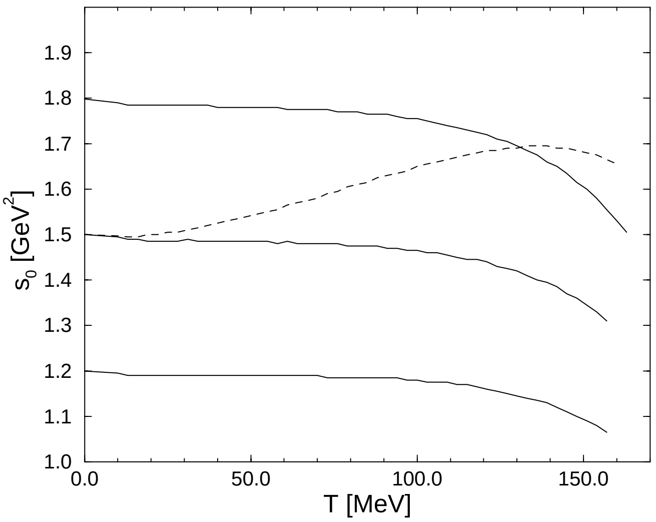

Fig. 2. Temperature evolution of the pQCD threshold s_0. The solid lines correspond to the case (b). For $s_0(0) = 1.5$ GeV^2 the case (a) is associated with the dashed line

greater effect on the T-evolution of m_ρ and s_0 than the logarithmic corrections in λ^2. For $s_0(0) = 1.5$ GeV^2 the behavior in the case (b) is qualitatively similar to that found in [2]. However, the influence of $s_0(T)$ on the sum rule is quite different in our work. Besides the truncation of the hadronic part of the spectral function, $s_0(T)$ also scales the vacuum part of the Gibbs averages in the OPE. In the case (a) we obtain an increase of $s_0(T)$ and hence a rise of the gluon condensate with increasing temperature which is in contradiction to the results obtained on the lattice and by effective meson models [8–10].

In Fig. 3 we finally show the temperature evolution of the gluon condensate normalized to its $T = 0$ value which due to the scaling relation of (29), the invariance under

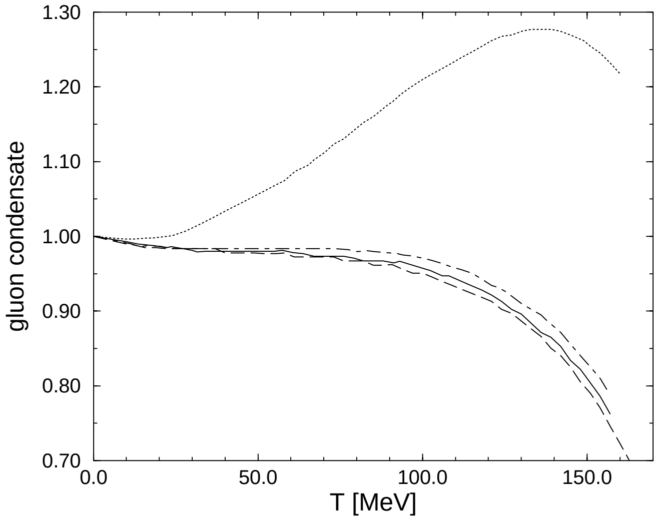

Fig. 3. Temperature evolution of the gluon condensate normalized to its zero temperature value. For the case (b) we take $s_0(0) = 1.2$ GeV^2, $s_0(0) = 1.5$ GeV^2, and $s_0(0) = 1.8$ GeV^2 corresponding to dot-dashed, solid, and long dashed lines, respectively. For $s_0(0) = 1.5$ GeV^2 the case (a) is associated with a dotted line

a change of the renormalization point, and the explicit T-independence in the chiral limit (according to (29)) is proportional to $s_0(T)^2$. We indicate the same combinations of initial values $s_0(0)$ and cases (a) and (b) as for the T-evolution of s_0. Figure 3 indicates a universality of the T evolution in a sense, that for different values of $s_0(0)$ almost identical results are obtained.

7 Summary and discussion

The main concern of this paper was the investigation of the consequences of a temperature dependent vacuum for the T-evolution of the ρ-meson mass and the gluon condensate in the chiral limit. Thereby, we used the method of thermal QCD sum rules. The results obtained in [2,6] for the thermal Operator Product Expansion containing also O(3) invariant contributions and for the thermal ρ^0 spectral function have been combined in our calculations. Following [2], the Gibbs averages of local, gauge invariant operators contributing to the thermal OPE of the ρ^0 current-current correlator were saturated by vacuum and one-pion matrix elements. For lack of better knowledge as far as the T-dependence of pionic twist two matrix elements is concerned we had to work with the $T = 0$ parton distributions of [21]. Consistency then demanded the use of a T-independent f_π, although the T-dependence of this quantity is known for low temperatures from chiral perturbation theory [22].

In contrast to previous sum rule calculations we suggested to account for a temperature dependence of the vacuum part of the Gibbs average. This was motivated by the observation, that the gluon condensate remains practically T-independent up to temperatures of 200 MeV in the sum rule analysis of [2], while a lattice measurement yields a drastic decrease at a critical temperature T_c of about $T_c \approx 140$ MeV [8]. To resolve this contradiction, we derived a scaling relation for the thermal vacuum average of scalar operators with the T-dependent spectral pQCD continuum threshold s_0. Thereby, the chiral limit, temperatures considerably smaller than the QCD scale Λ, and a renormalization point of the order of 1 GeV were assumed. The implementation of the above scaling relation is trivial for renormalization group invariant operators and becomes rather involved for operators which mix and scale anomalously under a change of the renormalization point. On the spectral side we used a narrow width approximation for the ρ^0 resonance since including a finite width Γ leads to ambiguities in the determination of m_ρ and Γ as was shown in [3] for the case of finite density. One would expect, that this is also true for finite temperatures. On the other hand, the effect of the scaling of the vacuum averages can only be isolated if one compares the results with previous calculations also using the narrow width approximation. In analogy to a previous sum rule calculation in the ρ^0-channel (see [6]) we employed a thermal pion continuum for the hadronic part, while the high energy tail was approximated by a T-independent pQCD continuum.

As compared to the vacuum case our sum rule calculations indicate a rise of the ρ-meson mass (17.5% at T_c) and a decrease of s_0 and of the gluon condensate with temperature. Thereby, the critical temperature T_c, where the sum rule analysis breaks down, is $T_c \approx 160$ MeV. This "critical" temperature is rather close to the QCD scale Λ, and therefore we do not expect the scaling relation of (29) to be a good approximation for the thermal vacuum average of scalar operators. The increase of the ρ-meson mass contradicts the sum rule results obtained in [1,2,23], where a monotonic decrease of this quantity is obtained. In the difference sum rule analysis of [6] the ρ-meson mass was also found to increase slightly ($\Delta m_\rho \approx$ 5 MeV) up to a temperature of about 125 MeV while decreasing thereafter. There are indications for a slight rise of the ρ-meson mass up to $T = 150$ MeV ($\Delta m_\rho \approx 30$ MeV at T=150 MeV) from a microscopical calculation of the spectral function in the ρ^0 channel using an effective $\rho - \pi$ Lagrangian [27]. We emphasize again, that the critical temperature of our calculation does only indicate the breakdown of the sum rule analysis based on a scaling relation for the scalar condensates and may not coincide with the critical value for a deconfinement phase transition. It is interesting to note, that a calculation in the Nambu-Jona-Lasinio model yields at $T \approx 160$ MeV and already at nuclear saturation density a relative increase of the dynamically generated constituent light-quark mass which is of the same order as that of the ρ-meson mass in our calculation [28,29]. In [15], where by means of PCAC, current algebra, and the LSZ reduction formula a mixing of the vector with the axial vector correlator for space-like q^2 was found for small temperatures (no OPE was used), a difference sum rule analysis yields an increase of the ρ-meson mass and a decrease of the mass of the a_1-meson with temperature. This was interpreted as the process of chiral symmetry restoration. The analysis of [15] is meaningful for small temperatures ($T < f_\pi$) and for a T independent f_π consistent with the recent lattice simulation of [19] indicating no sign of a T dependence up to $T \approx T_c$ of the chiral transition. Our result for the T-evolution of the ρ-meson mass is consistent with the one obtained in the analysis of [15]. In addition, qualitative arguments based on the instanton model [30] imply a cancellation of the effects of a decreasing constituent quark mass and a lowering of the instanton mediated attraction between constituent quarks for the T-evolution of the ρ-meson mass.

Depending on $s_0(0)$, our result for the gluon condensate exhibits a 25–30% decrease at $T \approx 160$ MeV which is compatible with the lattice data for two quark flavor QCD of [8]. The analysis showed, that the results are sensitive to the anomalous scaling of mass-dimension six operators under a change of the renormalization point (see Fig. 3). As Fig. 3 indicates the T-evolution of the gluon condensate is universal in a sense, that it is practically independent of the initial value $s_0(0)$.

There are still open questions concerning the inclusion of nonscalar, mixed operators at mass-dimension six in the thermal OPE. So far we must be content with the hope, that in the future their pionic matrix elements can be estimated from deep inelastic lepton scattering off the pion target [2].

To conclude, our analysis indicates, that in the Gibbs average the T-dependence of the vacuum matrix elements of scalar operators, as introduced via a scaling relation in s_0, produces a sizable decrease of the gluon condensate and a moderate increase of the ρ-meson mass up to temperatures of about 160 MeV, where the analysis breaks down (and the confidence in a good accuracy of the approximations made is not high anymore).

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Appendix A

Under the premises that T be considerably less than the fundamental QCD scale Λ, and that the renormalization scale Q_0 is taken of the order of 1 GeV we may argue in favor for the assumption of an implicit dependence of the scalar condensates on T via $s_0(T)$ in the chiral limit.

At $T = 0$ the most fundamental description of the QCD vacuum structure by the appearance of nonvanishing vacuum averages of local, scalar, and gauge invariant operators may in the chiral limit only involve the scale Λ and a renormalization scale Q_0. With the number of independent parameters kept fixed, we hypothesize that there

is a linear relation between Λ and $\sqrt{s_0(0)}$[7]. In this picture the constant of proportionality expresses this channel dependence. In the following we will argue that for $T > 0$ and provided that T is considerably smaller than Λ it is approximately possible to define a T-dependent fundamental scale Λ_T such that the thermal vacuum averages of scalar operators have the same functional dependence on Λ_T and Q_0 as in the case of $T = 0$, where they depend on Λ and Q_0. Then there should be the identical linear relation between Λ_T and $\sqrt{s_0(T)}$ as between Λ and $\sqrt{s_0(0)}$ at $T = 0$.
For the thermal vacuum average of $\mathcal{O}$ renormalized at Q_0 we may write

$$\langle 0|\mathcal{O}|0\rangle_T(Q_0) = f(T/\Lambda, T/Q_0, \Lambda/Q_0) \times Q_0^d \ , \qquad (A.1)$$

where f denotes a dimensionless function of its dimensionless arguments, and d indicates the mass dimension of $\mathcal{O}$. We define Λ_T by demanding

$$f(0, 0, \Lambda_T/Q_0) \stackrel{!}{=} f(T/\Lambda, T/Q_0, \Lambda/Q_0) \ . \qquad (A.2)$$

Solving (A.2) for Λ_T yields

$$\Lambda_T = \Lambda_T(\Lambda, Q_0, T) = \tilde{\Lambda}_T(T/\Lambda, T/Q_0, \Lambda/Q_0) \times \Lambda \ , \qquad (A.3)$$

where $\tilde{\Lambda}_T$ is a dimensionless function of its dimensionless arguments. Since in our applications the renormalization scale Q_0 is of the order of 1 GeV and with T considerably less than Λ, we can neglect the dependence of $\tilde{\Lambda}_T$ on T/Q_0

$$\Lambda_T(\Lambda, Q_0, T) \approx \bar{\Lambda}_T(T/\Lambda, \Lambda/Q_0) \times \Lambda \ , \qquad (A.4)$$

where $\bar{\Lambda}_T$ again is a dimensionless function of its dimensionless arguments. Let us now expand $\bar{\Lambda}_T$ in powers of Λ/Q_0

$$\Lambda_T = \Lambda \times (c_0(T/\Lambda) + c_1(T/\Lambda)\Lambda/Q_0 + \cdots) \ . \qquad (A.5)$$

Since $\Lambda_{T=0} \stackrel{!}{=} \Lambda$ it follows that

$$c_0(0) = 1 \quad c_i(0) = 0 \quad (i \geq 1) \ . \qquad (A.6)$$

Then the expansion of (A.5) should be well approximated by its first term if T is considerably less than Λ, namely by

$$\Lambda_T \approx \Lambda \times c_0(T/\Lambda) \ . \qquad (A.7)$$

Hence we obtain a well defined (since Q_0-independent) T-dependent scale Λ_T which yields the same functional dependence of the thermal vacuum average on Λ_T and Q_0 as that of the $T = 0$ vacuum average on Λ and Q_0.

Appendix B

Here we derive the decomposition of (21). For the operator $\mathcal{O}^s_{6,2}$ one simply uses

$$\bar{q}\gamma_\alpha t^a q = \bar{q}_L\gamma_\alpha t^a q_L + \bar{q}_R\gamma_\alpha t^a q_R$$

[7] There is then no other, independent scale to allow for a different behavior since s_0 cannot depend on the external variable Q_0.

to obtain

$$\mathcal{O}^s_{6,2} = \pi\alpha_s\,(P_4 + 2\ P_2) \ . \qquad (B.1)$$

Using

$$\bar{q}\gamma_\alpha\gamma_5 t^a q = \bar{q}_L\gamma_\alpha t^a q_L - \bar{q}_R\gamma_\alpha t^a q_R \ ,$$

yields the following decomposition for $\mathcal{O}^s_{6,1}$

$$\begin{aligned}\mathcal{O}^s_{6,1} = \pi\alpha_s\,\big[&(\bar\psi_L\gamma_\alpha t^a\tau^3\psi_L)^2 - \\ &2\bar\psi_L\gamma_\alpha t^a\tau^3\psi_L\bar\psi_R\gamma_\alpha t^a\tau^3\psi_R + \\ &(\bar\psi_R\gamma_\alpha t^a\tau^3\psi_R)^2\big] \ . \end{aligned}\qquad (B.2)$$

We are only interested in the flavor singlet part of the right-hand side, for which we also write $\mathcal{O}^s_{6,1}$

$$\begin{aligned}\mathcal{O}^s_{6,1} &= \frac{1}{3}\pi\alpha_s\,\big[(\bar\psi_L\gamma_\alpha t^a\tau^b\psi_L)^2 - \\ &\quad 2\bar\psi_L\gamma_\alpha t^a\tau^b\psi_L\bar\psi_R\gamma_\alpha t^a\tau^b\psi_R + \\ &\quad (\bar\psi_R\gamma_\alpha t^a\tau^b\psi_R)^2\big] \\ &= \frac{1}{3}\pi\alpha_s\,\big[(\bar\psi_L\gamma_\alpha t^a\tau^b\psi_L)^2 + \\ &\quad (\bar\psi_R\gamma_\alpha t^a\tau^b\psi_R)^2 - 2P_5\big] \ . \end{aligned}\qquad (B.3)$$

For the sake of brevity we only consider the following operator

$$\mathcal{O}_{L(R)} = (\bar\psi_{L(R)}\gamma_\alpha t^a\tau^b\psi_{L(R)})^2 \ . \qquad (B.4)$$

Writing out color and flavor indices (in this order), using

$$\tau^c_{ij}\tau^c_{mn} = 2\left(\delta_{in}\delta_{jm} - \frac{1}{N}\delta_{ij}\delta_{mn}\right) \ ,$$

with $N_c = 3$ and $N_f = 2$ one obtains

$$\begin{aligned}&\mathcal{O}_{L(R)} = \\ &4\Big[\bar\psi_{L(R),i\mu}\gamma_\alpha\delta_{im}\delta_{\mu\lambda}\psi_{L(R),j\nu}\bar\psi_{L(R),l\kappa}\gamma_\alpha\delta_{jl}\delta_{\nu\kappa}\psi_{L(R),m\lambda} - \\ &\frac{1}{2}\bar\psi_{L(R),i\mu}\gamma_\alpha\delta_{im}\delta_{\mu\nu}\psi_{L(R),j\nu}\bar\psi_{L(R),l\kappa}\gamma_\alpha\delta_{jl}\delta_{\kappa\lambda}\psi_{L(R),m\lambda} - \\ &\frac{1}{3}\bar\psi_{L(R),i\mu}\gamma_\alpha\delta_{ij}\delta_{\mu\lambda}\psi_{L(R),j\nu}\bar\psi_{L(R),l\kappa}\gamma_\alpha\delta_{lm}\delta_{\nu\kappa}\psi_{L(R),m\lambda} + \\ &\frac{1}{6}\bar\psi_{L(R),i\mu}\gamma_\alpha\delta_{ij}\delta_{\mu\nu}\psi_{L(R),j\nu}\bar\psi_{L(R),l\kappa}\gamma_\alpha\delta_{lm}\delta_{\kappa\lambda}\psi_{L(R),m\lambda}\Big] \ . \end{aligned}\qquad (B.5)$$

Applying the Fierz transformation [14]

$$\begin{aligned}&\bar\psi_{L(R),1}\gamma_\alpha\psi_{L(R),2}\bar\psi_{L(R),3}\gamma_\alpha\psi_{L(R),4} = \\ &\bar\psi_{L(R),1}\gamma_\alpha\psi_{L(R),4}\bar\psi_{L(R),3}\gamma_\alpha\psi_{L(R),2}\end{aligned}\qquad (B.6)$$

to the first and the third line of (B.5) and relabeling $m \leftrightarrow j$ in the second line, the sum of the second and the third line of (B.5) reads

$$-\frac{5}{6}\bar\psi_{L(R),i}\gamma_\alpha\psi_{L(R),j}\bar\psi_{L(R),j}\gamma_\alpha\psi_{L(R),i} \ , \qquad (B.7)$$

where the summation over flavor indices is implicit. This can easily be rewritten as

$$-\frac{5}{12}\bar{\psi}_{L(R)}\gamma_\alpha t^a\psi\bar{\psi}_{L(R)}\gamma_\alpha t^a\psi_{L(R)} \\ -\frac{5}{18}\bar{\psi}_{L(R)}\gamma_\alpha\psi\bar{\psi}_{L(R)}\gamma_\alpha\psi_{L(R)} \ . \tag{B.8}$$

Putting everything together we finally obtain

$$\mathcal{O}^s_{6,1} = \frac{1}{3}\pi\alpha_s\left(-\frac{5}{3}P_4 + \frac{32}{9}P_3 - 2P_5\right) \ . \tag{B.9}$$

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